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Finite element method for generalized piezothermoelastic problems

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Abstract

This paper is concerned with the generalized piezothermoelastic problems using finite element method (FEM). The governing equations are solved directly in time-domain to minimize precision losses caused during Laplace transformation. The results reveal that the heat wave propagating in medium at a finite speed can be described. Breakdown of a linear temperature drop at the heat wave front which cannot be described by Fourier's law is observed. Furthermore, the high concentration of stress and electric intensity at the heat wave front due to the high temperature gradient has been newly found.

Keywords: Generalized piezothermoelasticity; Finite element method; Thermal shock; G–L theory; Thermal relaxation time

1. Introduction

In classical thermoelastic theory, the heat equation is of a parabolic type, predicting the infinite speed of heat propagation. To resolve the paradox of this infinite speed of the heat propagation, a generalized thermoelastic theory was proposed in 1960s by Lord and Shulman (1967), the so-called "L–S theory." In the theory, Fourier's law of heat conduction was replaced by Maxwell–Cattaneo's law that generalized the Fourier's law by introducing a relaxation time. Another theory, G–L theory was proposed by Green and Lindsay (1972), which modified constitutive equations by introducing two relaxation times. Distinct feature in G–L theory is that classical Fourier's law of heat conduction is not violated if a medium under consideration has the center of symmetry. The heat conduction equations of the both theories are of wave-type, ensuring the finite speed of the heat propagation, the so-called "second sound effect." Since its introduction, G–L theory has been widely used by many researchers. These include Erbay and Suhubi (1986) for a longitudinal wave propagation problem in a circular infinite cylinder and Sherief (1994) for a one-dimensional thermo-mechanical shock problem. In the latter, a state space method which was developed earlier for G–L theory by Sherief (1993) was used.

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Due to the direct and converse piezoelectric effect, piezoelectric materials have been used in communication and space technologies. Mindlin (1961) deduced the equations describing the small vibrations of piezoelectric plates. In his study, the coupling among deformation, temperature, and electric fields was included. Ashida et al. (1994a,b) proposed a general solution procedure for the stationary three-dimensional (Ashida and Tauchert, 1994) and two-dimensional (Ashida et al., 1994a,b) problems of the piezothermoelastic solids in Cartesian coordinates. They also used the general solution to solve axisymmetric problems in cylindrical coordinates. The general solution was expanded by Choi et al. (1995) for two-dimensional transient problems. The above mentioned studies were based on classical Fourier's law of heat conduction. When the time scale of interest is very short, e.g. 0.1 ps, the relaxation time and inertia effects may be significant and cannot be neglected.

Because of the mathematical complexity, research work that deals with the piezothermoelastic problems that employ the generalized thermoelastic theory is few. Among them, Kaliski (1965) proposed a generalized piezothermoelectricity theory based on Maxwell–Cattaneo–Vernotte's heat conduction law. Chandrasekharaiah (1988) applied G–L theory to piezoelectric materials. In his subsequent work (Chandrasekharaiah, 1988) he derived the generalized piezoelectric constitutive equations and L–S type of heat conduction. Ashida and Tauchert (2003) dealt with dynamic thermoelastic problems of a thin piezoelectric plate and a circular piezoelectric plate subject to axisymmetric surface heating based on L–S theory (Ashida and Tauchert, 2004). In their study, the thermal–mechanical and thermal–electric coupling effects were not considered. In He et al. (2002)'s recent study, G–L theory was used. A thermo-elastic-piezoelectric effect was coupled with two-dimensional generalized thermal shock problem using a hybrid Laplace transformation-finite element method.

In solving one-dimensional generalized thermoelastic problems, Laplace transformation is often used and the inverse Laplace transformation can be obtained analytically. Therefore, the approximate analytical results can be obtained. There exists a clear wave front in the temperature distribution and hence the wave effect of heat conduction can be described precisely (He et al., 2003). However, in two-dimensional problems, Fourier transformation followed by Laplace transformation is required. Furthermore, the inverse Laplace and Fourier transformations should be performed numerically due to the complexity associated with the governing expressions. The results obtained through the above calculations for the two-dimensional problems may not be as accurate as in one-dimensional problems. In addition, the temperature front does not evolve during the computation, implying that the wave effect of heat conduction cannot be captured correctly (He et al., 2002). Although Laplace transformation and finite element method (FEM) can be combined to solve the two-dimensional generalized piezothermoelastic problems, the wave effect of heat conduction cannot be captured as in the one-dimension problems due to the numerical inverse Laplace transformation. Therefore, it is thought that the precision of the results may be improved if the numerical inverse Laplace transformation is avoided.

In this paper, finite element governing equations are derived based on G–L theory to directly solve the two-dimensional equations in time domain without Laplace transformation. An infinite piezoelectric plate subject to thermal shock is considered using a direct method developed in this study. The temperature, displacement, stress, and electric intensity in the plate are numerically obtained.

2. Fundamental equations of generalized piezothermoelasticity

Neglecting body force, free charge, and internal heat generation, the equilibrium equations of piezothermoelastic problem are expressed as

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{1}$$

$$q_{i,i} = -T_0 \rho \dot{\eta} + b_i T \tag{2}$$

$$D_{i,i} = 0 \tag{3}$$

The typical linear constitutive equations of a piezoelectric material under G–L generalized thermoelasticity are

$$\begin{aligned} \sigma_{ij} &= c_{ijkl}\varepsilon_{kl} - e_{kij}E_k - a_{ij}(\theta + \tau_1\dot{\theta}) \\ D_i &= e_{ikl}\varepsilon_{kl} + p_{ik}E_k + d_i(\theta + \tau_1\dot{\theta}) \\ \rho\eta &= a_{kl}\varepsilon_{kl} + d_kE_k + c_E(\theta + \tau_2\dot{\theta}) - b_i\theta_{,i} \end{aligned} \tag{4}$$

Table 1
The notations used in this paper

σ_{ij}	Components of stress tensor	ε_{ij}	Components of strain tensor
D_i	Electric displacement	E_i	Components of electric intensity
θ	Temperature increment $\theta = T - T_0$	T, T_0	Temperature and reference temperature
η	Entropy density	ϕ	Electric potential
q_i	Components of heat flux	c_{ijkl}	Elastic constants
e_{kij}, b_i	Piezoelectric constants	p_{ik}	Dielectric constants
d_i	Pyroelectric constants	a_{ij}	Thermal modulus
k_{ij}	Coefficients of thermal conductivity	c_E	Specific heat at constant deformation
λ, μ	Lame's constants	ρ	Mass density

where τ_1 and τ_2 are the relaxation times. Note that when $\tau_1 = \tau_2 = 0$, the problem is reduced to the classical piezothermoelastic problem. Fourier's heat conduction law can be expressed in tensor notation as

$$q_i = -k_{ij}\theta_{,j} \tag{5}$$

The relations between strain (ε) and displacement (u) and between the electric intensity (E) and electric potential (ϕ) can be expressed as

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2 \quad \text{and} \quad E_i = -\phi_{,i} \tag{6}$$

Notice that in the above equations, a comma followed by a suffix denotes the material derivatives and a superposed dot denotes the derivative with respect to time. The notations used in the above equations are summarized in Table 1.

If a transversely isotropic piezoceramic plate polarized along the y-axis is considered, all of the fundamental equations can be written in component form. After the mathematical manipulation, the governing equations of displacement and electric potential are yielded as Eqs. (7) and (8). Temperature terms are also included in the equations due to the coupled constitutive equations.

$$\begin{gathered}
c_{11}\frac{\partial^2 u}{\partial x^2}+c_{44}\frac{\partial^2 u}{\partial y^2}+c_{55}\frac{\partial^2 u}{\partial z^2}+(c_{12}+c_{44})\frac{\partial^2 v}{\partial x\partial y}+(c_{13}+c_{55})\frac{\partial^2 w}{\partial x\partial z}+(e_{12}+e_{13})\frac{\partial^2 \phi}{\partial x\partial y}-a_{11}\left(\frac{\partial\theta}{\partial x}+\tau_1\frac{\partial^2\theta}{\partial\tau\partial x}\right)=\rho\frac{\partial^2 u}{\partial t^2}\\
(c_{12}+c_{44})\frac{\partial^2 u}{\partial x\partial y}+c_{44}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}\right)+c_{22}\frac{\partial^2 v}{\partial y^2}+(c_{13}+c_{44})\frac{\partial^2 w}{\partial z\partial y}+e_{13}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial z^2}\right)+e_{22}\frac{\partial^2 \phi}{\partial y^2}-a_{22}\left(\frac{\partial\theta}{\partial y}+\tau_1\frac{\partial^2\theta}{\partial t\partial y}\right)=\rho\frac{\partial^2 v}{\partial t^2}\\
(c_{55}+c_{13})\frac{\partial^2 u}{\partial x\partial z}+(c_{12}+c_{44})\frac{\partial^2 v}{\partial z\partial y}+c_{55}\frac{\partial^2 w}{\partial x^2}+c_{44}\frac{\partial^2 w}{\partial y^2}+c_{11}\frac{\partial^2 w}{\partial z^2}+(e_{13}+e_{12})\frac{\partial^2 \phi}{\partial z\partial y}-a_{11}\left(\frac{\partial\theta}{\partial z}+\tau_1\frac{\partial^2\theta}{\partial t\partial z}\right)=\rho\frac{\partial^2 w}{\partial t^2}
\end{gathered} \tag{7}$$

$$(e_{12}+e_{13})\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 w}{\partial z\partial y}\right)+e_{13}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}\right)-p_{11}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial z^2}\right)-p_{22}\frac{\partial^2 \phi}{\partial y^2}+d_2\left(\frac{\partial\theta}{\partial y}+\tau_1\frac{\partial^2\theta}{\partial t\partial y}\right)=0 \tag{8}$$

Combining Eqs. (2), (4c), and (5), the temperature governing equation is finally obtained as follows:

$$k_{11}\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial z^2}\right)+k_{22}\frac{\partial^2\theta}{\partial y^2}=T_0\left[a_{11}\left(\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 w}{\partial z\partial t}\right)+a_{22}\frac{\partial^2 v}{\partial y\partial t}-d_2\frac{\partial^2\phi}{\partial y\partial t}+c_E\left(\frac{\partial\theta}{\partial t}+\tau_2\frac{\partial^2\theta}{\partial t^2}\right)\right] \tag{9}$$

In Eq. (9), variables u, v, and w are the displacements in the x-axis, y-axis, and z-axis, respectively. Eq. (9) is of a wave type equation indicating that temperature is conducted through the medium at a finite speed. Therefore, the generalized piezothermoelastic theory may predict the wave effect of heat conduction. Even though the governing equations of the displacement, electric potential, and temperature may be obtained, the generalized piezothermoelastic problem can hardly be solved analytically due to the complexity in the coupled equations. It can only be solved using a numerical method, e.g. finite element method.

3. Finite element formulations

In finite element method (FEM), the displacement $\{u\}$, the electric potential ϕ, and the temperature increment θ can be expressed in terms of corresponding nodal values of the elements $\{u^e\}$, $\{\phi^e\}$, and $\{\theta^e\}$ as

$$\{u\} = [N_1]\{u^e\}, \quad \phi = [N_2]\{\phi^e\}, \quad \theta = [N_2]\{\theta^e\} \tag{10}$$

where $[N_1]$ and $[N_2]$ are the shape function. Note that because the electric potential and temperature are scalar, their shape functions are the same. The strain $\{\varepsilon\}$, electric intensity $\{E\}$, and temperature gradient $\{\theta'\}$ may be expressed as

$$\{\varepsilon\} = [B_1]\{u^e\}, \quad \{E\} = [B_2]\{\phi^e\}, \quad \{\theta'\} = [B_2]\{\theta^e\} \tag{11}$$

where $[B_1]$ and $[B_2]$ are derived from $[N_1]$ and $[N_2]$ according to Eq. (6). If the body force is neglected, the principle of virtual work for the generalized piezothermoelasticity yields,

$$\begin{aligned}&\int_V \left(\delta\{\varepsilon\}^{\mathrm{T}}\{\sigma\} - \delta\{E\}^{\mathrm{T}}\{D\} + \delta\{\theta'\}^{\mathrm{T}}\{q\} - \delta\theta\ \rho T_0\dot{\eta}\right)\mathrm{d}V \\ &\quad = -\int_V \delta\{u\}^{\mathrm{T}}\rho\{\ddot{u}\}\mathrm{d}V + \int_{A_\sigma}\delta\{u\}^{\mathrm{T}}\{\bar{f}\}\mathrm{d}A + \int_{A_w}\delta\phi\bar{w}\,\mathrm{d}A + \int_{A_q}\delta\theta\bar{q}\,\mathrm{d}A\end{aligned} \tag{12}$$

where $\{\bar{f}\}$, $\bar{w}$, and $\bar{q}$ are, respectively, the force, the charge density, and the heat flux acted on surface A_σ, A_w and A_q. Substitution of Eqs. (4), (10), and (11) into Eq. (12) leads to the finite element governing equations as Eq. (13)

$$\sum_{e=1}^{\mathrm{ne}}\left(\begin{bmatrix} M^e_{mm} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M^e_{\theta\theta}\end{bmatrix}\begin{Bmatrix}\ddot{u}^e \\ \ddot{\phi}^e \\ \ddot{\theta}^e\end{Bmatrix} + \begin{bmatrix} 0 & 0 & -C^e_{m\theta} \\ 0 & 0 & -C^e_{e\theta} \\ C^e_{\theta m} & -C^e_{\theta e} & C^e_{\theta\theta}\end{bmatrix}\begin{Bmatrix}\dot{u}^e \\ \dot{\phi}^e \\ \dot{\theta}^e\end{Bmatrix} + \begin{bmatrix} K^e_{mm} & K^e_{me} & -K^e_{m\theta} \\ -K^e_{em} & K^e_{ee} & -K^e_{e\theta} \\ 0 & 0 & K^e_{\theta\theta}\end{bmatrix}\begin{Bmatrix}u^e \\ \phi^e \\ \theta^e\end{Bmatrix} = \begin{Bmatrix}T^e_m \\ T^e_e \\ T^e_\theta\end{Bmatrix}\right) \tag{13}$$

where ne is the number of elements. The coefficients and load vectors in Eq. (13) are given in Eq. (14).

$$\begin{aligned}
&[M^e_{mm}] = \int_V [N_1]^{\mathrm{T}}\rho[N_1]\mathrm{d}V, && [M^e_{\theta\theta}] = \int_V T_0[N_2]c\tau_2\{N_2\}^{\mathrm{T}}\,\mathrm{d}V \\
&[C^e_{m\theta}] = \int_V \tau_1[B_1]^{\mathrm{T}}\{a\}\{N_2\}^{\mathrm{T}}\,\mathrm{d}V, && [C^e_{e\theta}] = \int_V \tau_1[B_2]^{\mathrm{T}}\{d\}\{N_2\}^{\mathrm{T}}\,\mathrm{d}V \\
&[C^e_{\theta m}] = T_0\int_V [N_2]\{a\}^{\mathrm{T}}[B_1]\mathrm{d}V, && [C^e_{\theta e}] = \mathrm{T}_0\int_V [N_2]\{d\}^{\mathrm{T}}[B_2]\mathrm{d}V \\
&[C^e_{\theta\theta}] = T_0\int_V \left[[B_2]^{\mathrm{T}}\{b\}\{N_2\}^{\mathrm{T}} + [N_2]c\{N_2\}^{\mathrm{T}} + [N_2]\{b\}^{\mathrm{T}}[B_2]\right]\mathrm{d}V && \\
&[K^e_{mm}] = \int_V [B_1]^{\mathrm{T}}[C][B_1]\mathrm{d}V, && [K^e_{me}] = \int_V [B_1]^{\mathrm{T}}[e][B_2]\mathrm{d}V \\
&[K^e_{m\theta}] = \int_V [B_1]^{\mathrm{T}}\{a\}\{N_2\}^{\mathrm{T}}\,\mathrm{d}V, && [K^e_{em}] = \int_V [B_2]^{\mathrm{T}}[e]^{\mathrm{T}}[B_1]\mathrm{d}V \\
&[K^e_{ee}] = \int_V [B_2]^{\mathrm{T}}[p][B_2]\mathrm{d}V, && [K^e_{e\theta}] = \int_V [B_2]^{\mathrm{T}}\{d\}\{N_2\}^{\mathrm{T}}\,\mathrm{d}V \\
&[K^e_{\theta\theta}] = \int_V [B_2]^{\mathrm{T}}[k][B_2]\mathrm{d}V, && \{T^e_m\} = \int_{A_\sigma}[N_1]^{\mathrm{T}}\{\bar{T}\}\mathrm{d}A \\
&\{T^e_e\} = -\int_{A_w}[N_2]\bar{w}\,\mathrm{d}A, && \{T^e_\theta\} = -\int_{A_q}[N_2]\bar{q}\,\mathrm{d}A
\end{aligned} \tag{14}$$

Here, $[M]$, $[C]$, and $[K]$ are the mass, damping, and stiffness matrices, respectively. $\{\,T^e_m \quad T^e_e \quad T^e_\theta\,\}^{\mathrm{T}}$ is the load vector associated with the boundary conditions.

In general, the governing equations are solved as follows (He et al., 2002, 2003). First, take Laplace transformation to Eq. (13), second, solve the equations in Laplace domain, and third, transform the solutions into time domain by inverse Laplace transformation. However, the solution is not as accurate as that obtained from the analytical analysis because of numerical inverse Laplace transformation which causes numerical errors. The wave front of heat conduction cannot be observed in the temperature distribution plots (He et al., 2002).

This calculation procedure is complicated and time consuming. Furthermore, the essential properties of the generalized thermoelasticity obtained from the method may not be described correctly. On the other hand, Eq.

(13) appears to be solved directly in time domain by direct integration methods, e.g. the Newmark method. Because of the particularity of the generalized thermoelasticity, there exists a breakdown (i.e. a sudden drop) in the temperature distribution at the heat wave front (Chandrasekharaiah, 1984). It is worth noting that the size of the finite elements should be fine enough to capture the large temperature gradient at the heat wave front during the calculation and the medium should be meshed again due to the movement of the heat with time. This direct solving method was successfully applied to the generalized thermoelastic problem by Tian et al. (2006). Although the generalized piezothermoelastic problem is more complicated than the generalized thermoelastic problem, it is thought that the problem can be solved directly in time domain. Since the time scale of interest is very short, e.g. 0.1 ps (10^{-13} s), it is instructive to use non-dimensional quantities for numerical convenience as shown in Eq. (14).

$$
\begin{aligned}
&x_i^* = c_1\eta_1 x_i, \quad u_i^* = c_1\eta_1 u_i, \quad t^*(\tau_i^*) = c_1^2\eta_1 t(\tau_i), \quad \phi^* = c_1\eta_1\delta_1\phi \\
&\theta^* = \frac{\theta}{T_0}, \quad \sigma_{ij}^* = \frac{\sigma_{ij}}{\mu}, \quad D_i^* = \frac{D}{\mu\delta_1}, \quad c_1 = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}, \quad \eta_1 = \frac{\rho c_E}{\kappa}
\end{aligned} \tag{15}
$$

where c_1 is the propagating velocity of the elastic wave in solid and δ_1 is the piezoelectric constant. For the brevity, the asterisk symbol of the non-dimensional variables is dropped off in the following.

4. Numerical results

In the present study, a thick piezoelectric plate under thermal shock is considered. Fig. 1 shows the plate which is assumed to be infinitely long in the x- and z-directions and $2l$ is the thickness. The central parts of the upper and lower plate surfaces with the width of $2a$ are subject to the thermal shock. It is also assumed that the temperature of the heated area is $\theta = \theta_0 H(a - |x|)H(t)$, where $H(\cdot)$ and θ_0 denote the Heaviside unit step function and the prescribed temperature, respectively. The initial state of the plate is assumed to be traction free. The y-axis in the problem coincides with the polarization direction of the piezoelectric plate and the origin is on the upper side of the plate as shown in Fig. 1. The region under consideration is

$$\Omega = \{(x, y, z) : -\infty < x < \infty,\ 0 < y < 2l,\ -\infty < z < \infty\}$$

Due to the geometry of the plate and loading conditions, the strain has conditions as follows:

$$\varepsilon_{33} = \varepsilon_{23} = \varepsilon_{31} = 0 \tag{16}$$

Now, the problem can be treated as a plane strain problem in the x–y plane. Since the plate and loads are symmetric with respect to the y–z plane and the central plane of the plate, only one quarter of the cross section perpendicular to the z-axis, i.e. $OCAB$, is considered. The initial conditions are

$$u = v = \theta = \phi = 0 \quad \text{and} \quad \dot{u} = \dot{v} = \dot{\theta} = \dot{\phi} = 0 \quad \text{at } t = 0 \tag{17}$$

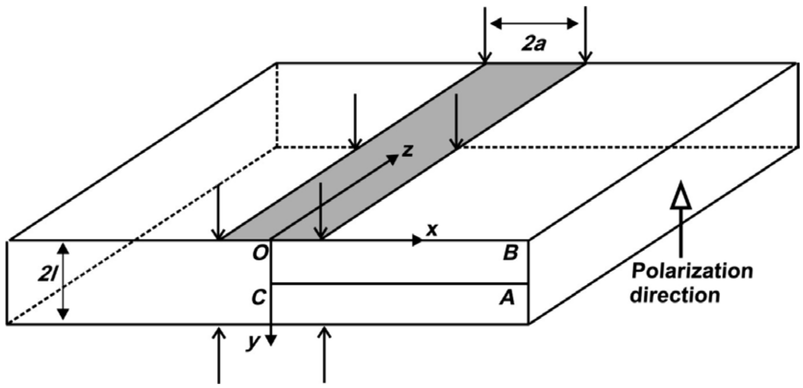


Fig. 1. Piezoelectric plate subject to thermal shock.

In the finite element analysis model, the length in the x-direction L_x is so large that the heat cannot reach edge AB during the analysis. This represents the infinite length of the plate in the x-direction. The boundary conditions are

$$\begin{aligned} &u = 0 \quad \text{on edge } OC(x = 0) \\ &u = v = \phi = 0 \quad \text{on edge } AB(x = L_x) \\ &v = \phi = 0 \quad \text{on edge } AC(y = l) \end{aligned} \tag{18}$$

To validate the present method, the plate material is considered to be cadmium selenide, having the following properties:

$$\begin{aligned} &c_{11} = 74.1 \times 10^9\ \mathrm{N\,m^{-2}}, && c_{12} = 45.2 \times 10^9\ \mathrm{N\,m^{-2}} \\ &c_{22} = 83.6 \times 10^9\ \mathrm{N\,m^{-2}}, && c_{44} = 13.2 \times 10^9\ \mathrm{N\,m^{-2}} \\ &e_{12} = -0.160\ \mathrm{C\,m^{-2}}, && e_{13} = -0.138\ \mathrm{C\,m^{-2}} \\ &e_{22} = 0.347\ \mathrm{C\,m^{-2}}, && p_{11} = 82.6 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}} \\ &p_{22} = 90.3 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}}, && a_{11} = 0.621 \times 10^6\ \mathrm{N\,K^{-1}\,m^{-2}} \\ &a_{22} = 0.551 \times 10^6\ \mathrm{N\,K^{-1}\,m^{-2}}, && K = 38\ \mathrm{kg\,m\,K^{-1}\,s^{-3}} \\ &d_2 = -2.94 \times 10^{-6}\ \mathrm{C\,K^{-1}\,m^{-2}}, && \delta_1 = -3.92 \times 10^{-12}\ \mathrm{C\,N^{-1}} \\ &\rho = 7600\ \mathrm{kg\,m^{-3}}, && c_E = 420\ \mathrm{J\,kg^{-1}\,K^{-1}} \end{aligned} \tag{19}$$

Based on the thermodynamical arguments, Green (1972) showed that the first relaxation time τ_1 is restricted by $\tau_1 \geqslant \tau_2 \geqslant 0$. In this analysis, the dimensionless relaxation times are set to be $\tau_1 = \tau_2 = 0.05$.

Fig. 2 shows the temperature contour in the area of $OCAB$. The white region represents "zero" temperature variation. The outmost contour is the position of the heat wave front. It is clear that the temperature gradient at the wave front along the y-axis is much larger than that along the x-axis. To show the temperature distribution along the y-axis and the x-axis, the non-dimensional temperatures along OC and OB are shown in Fig. 3(a) and (b), respectively. As expected, there is a sudden drop of the temperature in the y-axis. The non-dimensional temperature obtained by using Laplace transformation at $t = 0.20$ is also shown in Fig. 3(a) (He et al., 2002). Obviously, this feature cannot be observed if Laplace transformation is used. The location of the temperature drop appears to be consistent with the heat wave front. According to Eq. (9), the non-dimensional heat wave speed can be computed as

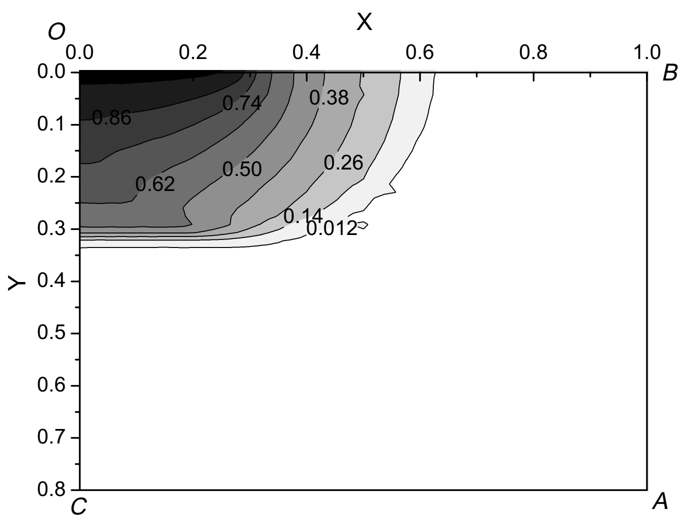


Fig. 2. The contour of temperature in $OBAC$ at the non-dimensional time $t = 0.07$.

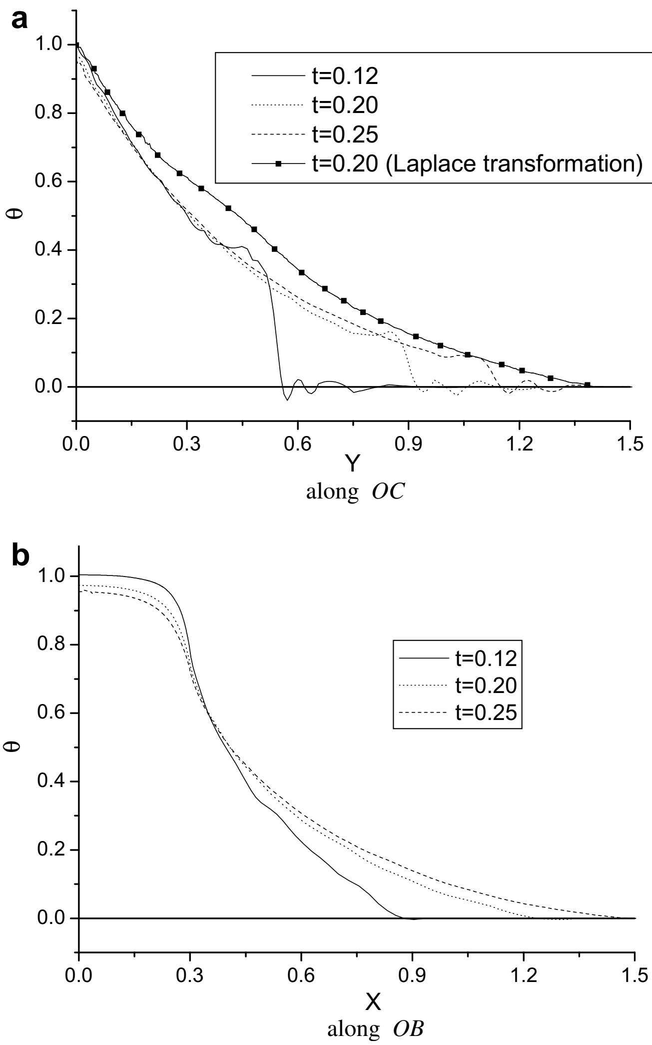


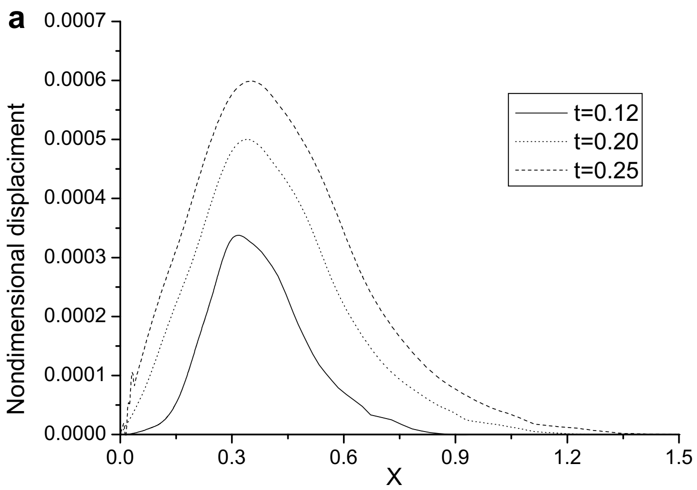
Fig. 3. Non-dimensional temperature profile along (a) OC and (b) OB at selected non-dimensional time.

$$(\sqrt{\tau_2})^{-1} = \left(\sqrt{0.05}\right)^{-1} = 4.47 \tag{20}$$

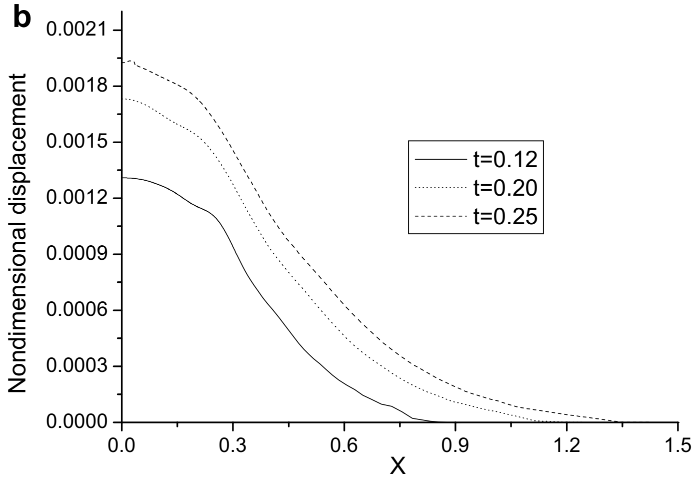
This indicates that at $t = 0.12$, 0.20, and 0.25, the heat wave front is located respectively at $y = 0.54$, 0.89, and 1.12. As shown in Fig. 3(a), the locations of the temperature drop (breakdown of Fourier's law) agree well with the theoretical prediction. It can be said that the developed method describes the wave effect of heat conduction accurately. Furthermore, the amplitude of the temperature drop decreases with time and this breakdown disappears after a long time. The latter implies that heat conduction now can be described by Fourier's law. In addition, because of the difference in the heat flux along the two directions, the temperature drop at the heat wave front in Fig. 3(b) is not prominent as that shown in Fig. 3(a).

Fig. 4(a) and (b) show the displacement along OB in the x- and the y-axes at the selected non-dimensional time. It can be seen from Fig. 4(a) that the displacement of the origin along the x-axis is zero. This is consistent with the symmetry condition. The displacement area affected becomes larger with time, indicating that the heat conducts through the plate at a finite speed.

Fig. 5 shows the displacement, the normal stress and electric intensity in the y-axis along OC. The displacement on the origin is largest and the direction is opposite to the y-axis. The normal stress on the origin is zero



in x-direction (u)



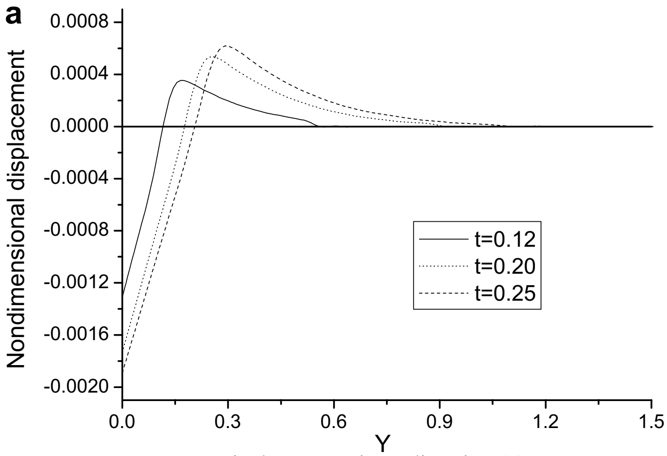
In y-direction (v)

Fig. 4. Non-dimensional displacement along OB at selected non-dimensional time.

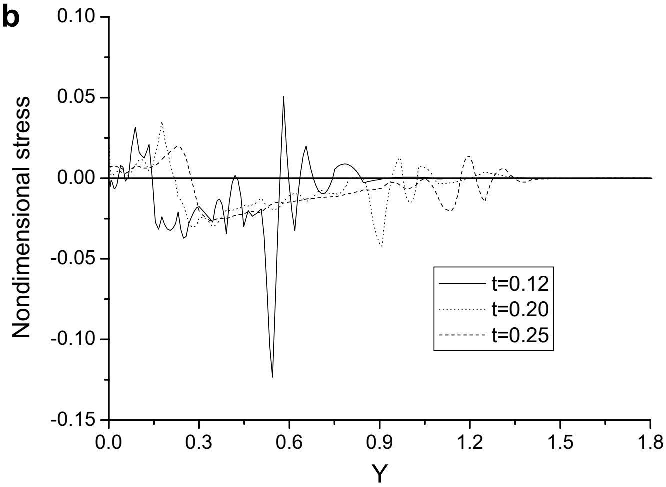
(traction free surface) and increases as moving away from the origin. After a certain distance, the stress drops suddenly having negative values. The location of the sudden change of its sign can be determined by the elastic wave speed. For example, the elastic wave front is at $y = 0.12$ when $t = 0.12$ due to the non-dimensional speed of the elastic wave is 1. This implies that the stress changes its sign at the position of $y = 0.12$ when $t = 0.12$. It is inferred that there exist the sudden drops in the stress and electric intensity. According to Fig. 5(b) and (c), it is concluded that the sudden drops of the stress and electric intensity occur at the heat wave front. These are caused by the large temperature gradient at the heat wave front. Again, it should be noted that these features in the stress and electric intensity cannot be observed if the transformation methods are used. Thus, it is evident that the direct solving method is more precise than the transformation methods.

5. Conclusions

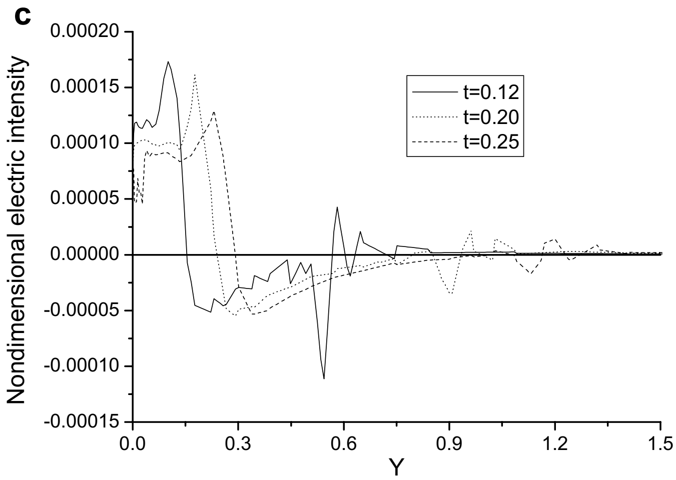
Two-dimensional generalized piezothermoelastic problem is presented in terms of Green and Lindsay generalized thermoelastic theory. Finite element method is used to directly solve the partial differential equations in time domain. The developed method has been applied to the generalized piezothermoelastic behavior of a



Displacement in y-direction (v)



Normal stress σ_{yy}



Induced electric intensity E_y

Fig. 5. Non-dimensional displacement, normal stress, and induced electric intensity along OC at selected non-dimensional time.

thick piezoelectric plate subject to the thermal shock. The results reveal that the method can precisely predict the wave type of heat conduction in a medium. The present solution procedure is capable of evaluating the breakdown of the normal stress and electric intensity at the heat wave front which cannot be observed if the conventional transformation methods are used. Therefore, the present method is promising to analyze the generalized piezothermoelastic problem effectively and precisely.

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